Mathematical Structures in Magnetohydrodynamics

Action Principles, Poisson Brackets, Symmetries and Conservation Laws

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Outline

1. Ideal Magnetohydrodynamics

2. Poisson Brackets

3. Variational Principles

4. Formal Lagrangians

Ideal Magnetohydrodynamics

Ideal Compressible Magnetohydrodynamics

- advective form

$$
\begin{aligned}
\rho v_t + \rho v \cdot \nabla v &= B \cdot \nabla B - \nabla p, \\
\rho_t + \nabla \cdot (\rho v) &= 0, \\
s_t + v \cdot \nabla s &= 0, \\
B_t - \nabla \times (v \times B) &= 0,
\end{aligned}
$$

v velocity field

ρ fluid density

s entropy per unit mass

B magnetic field strength

p gas pressure

ε internal energy

$$p = \rho^2 \varepsilon_\rho(\rho, s), \qquad \varepsilon = \beta \rho^{\gamma-1} \exp\{(\gamma - 1)/\alpha\, s\}, \qquad P = p + \tfrac{1}{2}\,|v|^2 - |B|^2$$

Ideal Compressible Magnetohydrodynamics

- advective form

$$\rho v_t + \rho v \cdot \nabla v = \cancel{B \cdot \nabla B} - \nabla p,$$
$$\rho_t + \nabla \cdot (\rho v) = 0,$$
$$s_t + v \cdot \nabla s = 0,$$
$$\cancel{B_t - \nabla \times (v \times B)} = 0,$$

v velocity field	s entropy per unit mass	p gas pressure
ρ fluid density	B magnetic field strength	ε internal energy

$$p = \rho^2 \varepsilon_\rho(\rho, s), \qquad \varepsilon = \beta \rho^{\gamma-1} \exp\{(\gamma-1)/\alpha\, s\}, \qquad P = p + \tfrac{1}{2}|v|^2 - |B|^2$$

Poisson Brackets

Hamiltonian Dynamics and Poisson Brackets

- let $u(t,x) = (u^1,\, u^2,\, ...,\, u^m)^T$ be the field variables of some system of partial differential equations, defined over the space Ω with coordinates x and $\mathcal{F}$ an arbitrary functional of the field variables u
- if the system is Hamiltonian the evolution of any such functional $\mathcal{F}$ is given by

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\}$$

- $\mathcal{H}$ is the Hamiltonian functional, usually the total energy of the system
- the Poisson bracket $\{\cdot,\cdot\}$ is an bilinear, anti-symmetric operator of the form

$$\{\mathcal{F}, \mathcal{G}\} = \int_{\Omega} \frac{\delta \mathcal{F}}{\delta u^i}\, \mathcal{J}^{ij}(u)\, \frac{\delta \mathcal{G}}{\delta u^j}\, \mathrm{d}x$$

  where $\mathcal{F}$ and $\mathcal{G}$ are functionals of u and $\delta\mathcal{F}/\delta u^i$ is the functional derivative

$$\frac{d}{d\epsilon}\mathcal{F}\big[u^1,\, ...,\, u^i + \epsilon v^i,\, ...,\, u^m\big]\Big|_{\epsilon=0} = \int_{\Omega} \frac{\delta \mathcal{F}}{\delta u^i}\, v^i\, \mathrm{d}x$$

Hamiltonian Dynamics and Poisson Brackets

- for Hamiltonian systems, the evolution of any functional $\mathcal{F}$ is given by

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\} \qquad \text{with} \qquad \{\mathcal{F}, \mathcal{G}\} = \int\limits_{\Omega} \frac{\delta \mathcal{F}}{\delta u^i} \, \mathcal{J}^{ij}(u) \, \frac{\delta \mathcal{G}}{\delta u^j} \, \mathrm{d}x$$

- the Poisson bracket $\{\cdot, \cdot\}$ satisfies Leibniz' rule and the Jacobi identity

$$\{\{\mathcal{F}, \mathcal{G}\}, \mathcal{H}\} + \{\{\mathcal{G}, \mathcal{H}\}, \mathcal{F}\} + \{\{\mathcal{H}, \mathcal{F}\}, \mathcal{G}\} = 0$$

  for arbitrary functionals $\mathcal{F}, \mathcal{G}, \mathcal{H}$ of u

- $\mathcal{J}(u)$ is an anti-self-adjoint operator, which has the property that

$$\sum_{l=1}^{m} \left(\frac{\partial \mathcal{J}^{ij}(u)}{\partial u^l} \, \mathcal{J}^{lk}(u) + \frac{\partial \mathcal{J}^{jk}(u)}{\partial u^l} \, \mathcal{J}^{li}(u) + \frac{\partial \mathcal{J}^{ki}(u)}{\partial u^l} \, \mathcal{J}^{lj}(u) \right) = 0$$

  for $1 \leq i, j, k \leq m$, ensuring that the bracket $\{\cdot, \cdot\}$ satisfies the Jacobi identity

- apart from that, $\mathcal{J}(u)$ is not required to be of any particular form and is allowed to depend on the fields u in an arbitrarily complicated way (nonlinear, differential and integral operators)

Hamiltonian Dynamics and Poisson Brackets

- for Hamiltonian systems, the evolution of any functional $\mathcal{F}$ is given by

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\} \qquad \text{with} \qquad \{\mathcal{F}, \mathcal{G}\} = \int\limits_{\Omega} \frac{\delta \mathcal{F}}{\delta u^i} \, \mathcal{J}^{ij}(u) \, \frac{\delta \mathcal{G}}{\delta u^j} \, \mathrm{d}z$$

- Hamiltonian systems preserve energy due to anti-symmetry of the Poisson bracket

$$\frac{d\mathcal{H}}{dt} = \{\mathcal{H}, \mathcal{H}\} = -\{\mathcal{H}, \mathcal{H}\} = 0$$

- if the Hamiltonian is constant along the flow of some functional Φ, i.e., $\{\mathcal{H}, \Phi\} = 0$, then Φ is a momentum map that is preserved by the flow of $\mathcal{H}$ as

$$\frac{d\Phi}{dt} = \{\Phi, \mathcal{H}\} = -\{\mathcal{H}, \Phi\} = 0$$

- if $\mathcal{J}(u)$ has a non-empty nullspace, there exist so-called Casimir invariants, that is functionals $\mathcal{C}$ for which $\{\mathcal{F}, \mathcal{C}\} = 0$ for all functionals $\mathcal{F}$, i.e.,

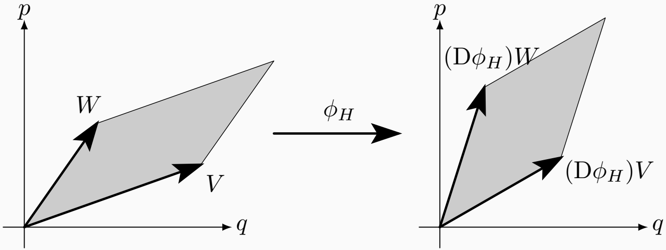
$$\mathcal{J}^{ij}(u) \, \frac{\delta \mathcal{C}}{\delta u^j} = 0$$

Hamiltonian Dynamics and Poisson Brackets: What does all of that mean?

- phase space circulation theorem (similar to ordinary fluids): conservation of loop integrals along any closed curve Γ in phasespace

$$J = \oint_{\Gamma} p \cdot \mathrm{d}q \qquad \text{with} \qquad \frac{d}{dt} J = 0$$

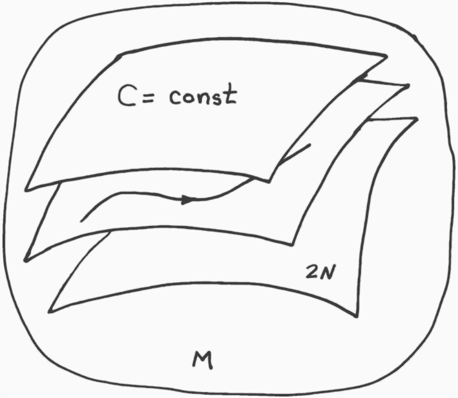
- symplecticity: conservation of phasespace area (and as consequence of phasespace volume)



- analogously conservation of higher-order Poincaré invariants (in total 2N invariants: "loop integrals" of dimension $1, 3, 5, ..., 2N-1$ and surfaces of dimension $2, 4, 6, ..., 2N$)

Hamiltonian Dynamics and Poisson Brackets: What does all of that mean?

- local structure of a Poisson manifold



- phasespace is foliated by the level sets of the Casimir invariants
- every orbit remains on the surface defined by the initial values of the Casimir invariants

Poisson Brackets for Compressible Magnetohydrodynamics

- compressible magnetohydrodynamics

$$\{\mathcal{F},\mathcal{G}\}[\rho,v,s,B] = -\int \bigg[\frac{\delta\mathcal{F}}{\delta\rho}\nabla\cdot\frac{\delta\mathcal{G}}{\delta v} - \frac{\delta\mathcal{G}}{\delta\rho}\nabla\cdot\frac{\delta\mathcal{F}}{\delta v} + \frac{\delta\mathcal{F}}{\delta v}\cdot\left(\frac{\nabla\times v}{\rho}\times\frac{\delta\mathcal{G}}{\delta v}\right) + \frac{1}{\rho}\nabla s\cdot\left(\frac{\delta\mathcal{F}}{\delta s}\frac{\delta\mathcal{G}}{\delta v} - \frac{\delta\mathcal{G}}{\delta s}\frac{\delta\mathcal{F}}{\delta v}\right)$$
$$+ \frac{1}{\rho}\frac{\delta\mathcal{F}}{\delta v}\cdot\left(B\times\left(\nabla\times\frac{\delta\mathcal{G}}{\delta B}\right)\right) + \frac{\delta\mathcal{F}}{\delta B}\cdot\left(\nabla\times\left(B\times\frac{1}{\rho}\frac{\delta\mathcal{G}}{\delta v}\right)\right)\bigg]\,\mathrm{d}x$$

$$\mathcal{H} = \int \left[\tfrac{1}{2}\rho\,|v|^2 + \rho\varepsilon(\rho,s) + \tfrac{1}{2}\,|B|^2\right]\mathrm{d}x$$

- momentum variables: $m = \rho v$, $\sigma = \rho s$

$$\{\mathcal{F},\mathcal{G}\}[\rho,m,\sigma,B] = -\int \bigg[\rho\left(\frac{\delta\mathcal{F}}{\delta m}\nabla\cdot\frac{\delta\mathcal{G}}{\delta\rho} - \frac{\delta\mathcal{G}}{\delta m}\nabla\cdot\frac{\delta\mathcal{F}}{\delta\rho}\right) + m\cdot\left(\frac{\delta\mathcal{F}}{\delta m}\cdot\nabla\frac{\delta\mathcal{G}}{\delta m} - \frac{\delta\mathcal{G}}{\delta m}\cdot\nabla\frac{\delta\mathcal{F}}{\delta m}\right)$$
$$+ \sigma\left(\frac{\delta\mathcal{F}}{\delta m}\nabla\cdot\frac{\delta\mathcal{G}}{\delta\sigma} - \frac{\delta\mathcal{G}}{\delta m}\nabla\cdot\frac{\delta\mathcal{F}}{\delta\sigma}\right) + B\cdot\left(\frac{\delta\mathcal{F}}{\delta m}\nabla\cdot\frac{\delta\mathcal{G}}{\delta B} - \frac{\delta\mathcal{G}}{\delta m}\nabla\cdot\frac{\delta\mathcal{F}}{\delta B}\right)$$
$$+ \left(\nabla\frac{\delta\mathcal{F}}{\delta B}\right)\cdot\frac{\delta\mathcal{G}}{\delta m} - \left(\nabla\frac{\delta\mathcal{G}}{\delta B}\right)\cdot\frac{\delta\mathcal{F}}{\delta m}\bigg]\,\mathrm{d}x$$

$$\mathcal{H} = \int \left[\tfrac{1}{2}\rho^{-1}\,|m|^2 + \rho\varepsilon(\rho,\sigma) + \tfrac{1}{2}\,|B|^2\right]\mathrm{d}x$$

Poisson Brackets for Compressible Fluids

- compressible Euler equation

$$\{\mathcal{F}, G\}[v,\rho,s] = -\int \left[\frac{\delta\mathcal{F}}{\delta\rho}\nabla\cdot\frac{\delta G}{\delta v} - \frac{\delta G}{\delta\rho}\nabla\cdot\frac{\delta\mathcal{F}}{\delta v} + \frac{\delta\mathcal{F}}{\delta v}\cdot\left(\frac{\nabla\times v}{\rho}\times\frac{\delta G}{\delta v}\right) + \frac{1}{\rho}\nabla s\cdot\left(\frac{\delta\mathcal{F}}{\delta s}\frac{\delta G}{\delta v} - \frac{\delta G}{\delta s}\frac{\delta\mathcal{F}}{\delta v}\right)\right]\mathrm{d}x$$

$$\mathcal{H} = \int \left[\tfrac{1}{2}\rho\,|v|^2 + \rho\varepsilon(\rho,s)\right]\mathrm{d}x$$

- momentum variables: $m = \rho v$, $\sigma = \rho s$

$$\{\mathcal{F},\mathcal{G}\}[\rho,m,\sigma] = -\int \Bigg[\rho\left(\frac{\delta\mathcal{F}}{\delta m}\nabla\cdot\frac{\delta\mathcal{G}}{\delta\rho} - \frac{\delta\mathcal{G}}{\delta m}\nabla\cdot\frac{\delta\mathcal{F}}{\delta\rho}\right) + m\cdot\left(\frac{\delta\mathcal{F}}{\delta m}\cdot\nabla\frac{\delta\mathcal{G}}{\delta m} - \frac{\delta\mathcal{G}}{\delta m}\cdot\nabla\frac{\delta\mathcal{F}}{\delta m}\right) + \sigma\left(\frac{\delta\mathcal{F}}{\delta m}\nabla\cdot\frac{\delta\mathcal{G}}{\delta\sigma} - \frac{\delta\mathcal{G}}{\delta m}\nabla\cdot\frac{\delta\mathcal{F}}{\delta\sigma}\right)\Bigg]\mathrm{d}x$$

$$\mathcal{H} = \int \left[\tfrac{1}{2}\rho^{-1}\,|m|^2 + \rho\varepsilon(\rho,\sigma)\right]\mathrm{d}x$$

- Burgers' equation

$$\{\mathcal{F},\mathcal{G}\}[v] = -\int v\left(\frac{\delta\mathcal{F}}{\delta v}\frac{\partial}{\partial x}\frac{\delta\mathcal{G}}{\delta v} - \frac{\delta\mathcal{G}}{\delta v}\frac{\partial}{\partial x}\frac{\delta\mathcal{F}}{\delta v}\right)\mathrm{d}x$$

$$\mathcal{H} = \frac{1}{2}\int |v|^2\,\mathrm{d}x$$

Discretisation of Poisson Brackets: Why and how?

Why?

- preserving anti-symmetry immediately leads to energy preserving algorithms (easy!)
- preserving Casimir invariants leads to conservation law preserving algorithms (not too hard)
- preserving the Jacobi identity leads to phasespace structure and Poincaré invariant preserving algorithms (very hard!)

But how?

- constant Poisson structure (e.g. Maxwell): anything goes (only antisymmetry required!)
- GEMPIC: finite element particle-in-cell methods for Vlasov–Maxwell
- grid-based methods: .... open problem!
- magnetohydrodynamics: .... open problem!

Discretisation of Poisson Brackets: GEMPIC

- particle-like distribution function for N_p particles labeled by a,

$$f_h(x, v, t) = \sum_{a=1}^{N_p} w_a \, \delta\big(x - x_a(t)\big) \, \delta\big(v - v_a(t)\big),$$

  with weights w_a, particle positions x_a and particle velocities v_a

- 1-form and 2-form spline or finite element basis functions (vector-valued)

$$\Lambda^1_\alpha(x) = \begin{pmatrix} \Lambda^{1,1}_\alpha(x) \\ \Lambda^{1,2}_\alpha(x) \\ \Lambda^{1,3}_\alpha(x) \end{pmatrix}, \qquad \Lambda^2_\alpha(x) = \begin{pmatrix} \Lambda^{2,1}_\alpha(x) \\ \Lambda^{2,2}_\alpha(x) \\ \Lambda^{2,3}_\alpha(x) \end{pmatrix}$$

- semi-discrete electric field E_h and magnetic field B_h with coefficient vectors e and b

$$E_h(t, x) = \sum_{\alpha=1}^{N_{\text{dof}}} e_\alpha(t) \, \Lambda^1_\alpha(x), \qquad B_h(t, x) = \sum_{\alpha=1}^{N_{\text{dof}}} b_\alpha(t) \, \Lambda^2_\alpha(x)$$

Discretisation of Poisson Brackets: GEMPIC

- semi-discrete degrees of freedom $[\mathbf{X}, \mathbf{V}, \mathbf{e}, \mathbf{b}]$
- semi-discrete Poisson bracket $\{F, G\}_d$ (anti-symmetric, Jacobi identity, preserves Casimirs)
- semi-discrete Hamiltonian $\hat{\mathcal{H}} = \frac{1}{2}\,\mathbf{V}^\top \mathbb{M}_p \mathbf{V} + \frac{1}{2}\,\mathbf{e}^\top M_1 \mathbf{e} + \frac{1}{2}\,\mathbf{b}^\top M_2 \mathbf{b}$
- semi-discrete equations of motion

$$
\begin{aligned}
\dot{\mathbf{X}} &= \{\mathbf{X}, \hat{\mathcal{H}}\}_d = \mathbf{V}, & \frac{dx_s}{dt} &= v_s, \\
\dot{\mathbf{V}} &= \{\mathbf{V}, \hat{\mathcal{H}}\}_d = \mathbb{M}_p^{-1} \mathbb{M}_q \big(\mathbb{\Lambda}^1(\mathbf{X}) \mathbf{e} + \mathbb{B}(\mathbf{X}, \mathbf{b}) \mathbf{V} \big), & \frac{dv_s}{dt} &= e_s \big(E(x_s) + v_s \times B(x_s) \big), \\
\dot{\mathbf{e}} &= \{\mathbf{e}, \hat{\mathcal{H}}\}_d = \mathbb{M}_1^{-1} \big(\mathbb{C}^\top \mathbb{M}_2 \mathbf{b} - \mathbb{\Lambda}^1(\mathbf{X})^\top \mathbb{M}_q \mathbf{V} \big), & \frac{\partial E}{\partial t} &= \operatorname{curl} B - J, \\
\dot{\mathbf{b}} &= \{\mathbf{b}, \hat{\mathcal{H}}\}_d = -\mathbb{C} \mathbf{e}, & \frac{\partial B}{\partial t} &= -\operatorname{curl} E
\end{aligned}
$$

- time integration: splitting methods or discrete gradients

Discretisation of Poisson Brackets: GEMPIC for MHD or Euler?

- starting point: Euler bracket in momentum variables

$$\{\mathcal{F},\mathcal{G}\}[\rho,m,\sigma] = -\int \Big[\rho\Big(\frac{\delta\mathcal{F}}{\delta m}\nabla\cdot\frac{\delta\mathcal{G}}{\delta\rho} - \frac{\delta\mathcal{G}}{\delta m}\nabla\cdot\frac{\delta\mathcal{F}}{\delta\rho}\Big) + m\cdot\Big(\frac{\delta\mathcal{F}}{\delta m}\cdot\nabla\frac{\delta\mathcal{G}}{\delta m} - \frac{\delta\mathcal{G}}{\delta m}\cdot\nabla\frac{\delta\mathcal{F}}{\delta m}\Big) + \sigma\Big(\frac{\delta\mathcal{F}}{\delta m}\nabla\cdot\frac{\delta\mathcal{G}}{\delta\sigma} - \frac{\delta\mathcal{G}}{\delta m}\nabla\cdot\frac{\delta\mathcal{F}}{\delta\sigma}\Big)\Big]\,\mathrm{d}x$$

- particle-like density, velocity, momentum and entropy for N_p particles labeled by a,

$$\rho_h(x,t) = \sum_{a=1}^{N_p} w_a\, m_a\, \delta\big(x - x_a(t)\big), \qquad \sigma_h(x,t) = \sum_{a=1}^{N_p} w_a\, m_a\, \sigma_a\, \delta\big(x - x_a(t)\big),$$

$$v_h(x,t) = \sum_{a=1}^{N_p} w_a\, v_a\, \delta\big(x - x_a(t)\big), \qquad m_h(x,t) = \sum_{a=1}^{N_p} w_a\, m_a\, v_a\, \delta\big(x - x_a(t)\big)$$

with weights w_a, mass m_a, particle positions x_a, particle velocities v_a and particle entropies σ_a

Discretisation of Poisson Brackets: GEMPIC for MHD or Euler?

- particle-like density, velocity, etc.

$$\rho_h(x,t) = \sum_{a=1}^{N_p} w_a \, m_a \, \delta\big(x - x_a(t)\big), \qquad v_h(x,t) = \sum_{a=1}^{N_p} w_a \, v_a \, \delta\big(x - x_a(t)\big)$$

- Hamiltonian and internal energy

$$\mathcal{H} = \int \left[\tfrac{1}{2} v \cdot m + \rho \varepsilon(\rho, \sigma) \right] dx, \qquad \varepsilon = \beta \rho^{\gamma - 1} \exp\{(\gamma - 1)/\alpha \, s\}$$

- requires continuous (grid-based or r.b.f.) representations of density, velocity, etc., e.g.,

$$\bar{\rho}_h(x,t) = \sum_{i=1}^{N} \rho_i(t) \, \phi_i(x), \qquad \bar{v}_h(x,t) = \sum_{i=1}^{N} v_i(t) \, \psi_i(x)$$

- coefficients determined e.g. by L^2 projection

$$\int \bar{v}_h(x,t) \, \psi_i(x) \, dx = \int v_h(x,t) \, \psi_i(x) \, dx \qquad \Rightarrow \qquad M_{ij} v_j = \sum_{a=1}^{N_p} w_a v_a \, \psi_i(x_a(t))$$

Discretisation of Poisson Brackets: Eulerian Methods?

- momentum map and Casimir conservation: many schemes work (FD, FV, FEM, DG, ...)
- Jacobi identity: under investigation
  - best hope: finite volume and discontinuous Galerkin spectral element methods
  - current status: Jacobi identity is preserved separately for the element and jump discretisation but there are residual terms for the full scheme
- BUT: the construction of split-forms and energy preserving schemes become automatic

$$\{\mathcal{F},\mathcal{G}\}[v] = -\int_\Omega v(x') \left(\frac{\delta \mathcal{F}}{\delta v} \frac{\partial}{\partial x'} \frac{\delta \mathcal{G}}{\delta v} - \frac{\delta \mathcal{G}}{\delta v} \frac{\partial}{\partial x'} \frac{\delta \mathcal{F}}{\delta v} \right) \mathrm{d}x', \qquad \mathcal{H} = \frac{1}{2} \int_\Omega |v(x)|^2 \, \mathrm{d}x,$$

$$v_t(x) = \{v, H\} = -\int_\Omega v(x') \left(\delta(x - x') \frac{\partial}{\partial x'} v(x') - v(x') \frac{\partial}{\partial x'} \delta(x - x') \right) \mathrm{d}x'$$

$$= -\int_\Omega \left(v(x') \frac{\partial}{\partial x'} v(x') + \frac{\partial}{\partial x'} v(x')^2 \right) \delta(x - x') \, dx' + \int_{\partial\Omega} v(x')^2 \delta(x - x') \, \mathrm{d}x'$$

$$0 = v_t(x) + v(x)\, v_x(x) + (v(x)^2)_x - \left[v(x)^2 \right]_{\partial\Omega}$$

Variational Principles

Eulerian Variational Principles for Fluids

- Eulerian action for compressible magnetohydrodynamics

$$\mathcal{A}[v, \rho, s, B] = \int_0^T \int_\Omega \left[\frac{1}{2} \rho \left| v \right|^2 - \rho \varepsilon(\rho, s) - \frac{1}{2} \left| B \right|^2 \right] \mathrm{d}x \mathrm{d}t$$

- constraint variations

$$\delta v = u_t + [v, u] = u_t + v \cdot \nabla u - u \cdot \nabla v, \qquad \delta \rho = - \pounds_u \rho = - \nabla \cdot (\rho u),$$

$$\delta B = - \pounds_u B = \nabla \times (u \times B), \qquad \delta s = - \pounds_u s = - u \cdot \nabla s$$

- requires discretisation of the underlying Lie algebra structure
- BUT: the construction of split-forms becomes automatic

$$\delta \mathcal{A} = \delta \int_0^T \int_\Omega \frac{1}{2} \left| v \right|^2 \, \mathrm{d}x \mathrm{d}t = \int_0^T \int_\Omega v \, \delta v(x) \, \mathrm{d}x \mathrm{d}t = \int_0^T \int_\Omega v \left[u_t + v u_x - u v_x \right] \mathrm{d}x \mathrm{d}t$$

$$= - \int_\Omega u \left[v_t + (v^2)_x + v v_x \right] \mathrm{d}x + \left[\int_\Omega v u \, dx \right]_0^T + \int_0^T \int_{\partial \Omega} v^2 u \, \mathrm{d}x \mathrm{d}t = 0 \text{ for all } u$$

Formal Lagrangians

Nonvariational PDEs and Formal Lagrangians

- fluid dynamics and plasma physics: equations of advection-diffusion type

$$u_t + \nabla \cdot (fu) = C[u]$$

→ no natural eulerian action principles (except Euler-Poincaré)

- treat nonvariational systems as part of larger Lagrangian systems

$$\Big(u, F[u] = 0\Big) \qquad \rightarrow \qquad \Big(u, v, F[u] = 0, F^{*}[u, v] = 0\Big)$$

- formal action of the extended system of differential equations

$$\mathcal{A}[u, v] = \langle F[u], v \rangle = \int L \, \mathrm{d}t \, \mathrm{d}x \qquad \text{with formal Lagrangian} \qquad L = v \cdot F[u]$$

- variational principle: original and adjoint equation

$$\frac{\delta \mathcal{A}}{\delta v} = F[u] = 0, \qquad \frac{\delta \mathcal{A}}{\delta u} = F^{*}[u, v] = 0$$

→ use formal Lagrangians to derive variational integrators for arbitrary systems

Noether's Theorem and Formal Lagrangians

- consider a transformation of the fields $y = (u, v)$

$$y \to y^\epsilon(x) = \eta(x, y(x), \epsilon) \qquad \text{with} \qquad \eta\big|_{\epsilon=0} = \mathrm{id}$$

- symmetry: Lagrangian $L(x, y, Dy)$ is invariant under transformation

$$L\big(x, \eta, D\eta\big) = L\big(x, y, Dy\big)$$

- conservation law: divergence of the Noether current J vanishes

$$\mathrm{div}\, J = \frac{\partial}{\partial x^\mu}\left[\frac{\partial L}{\partial y^a_\mu}(x, y, Dy) \cdot V^a\right] = 0 \qquad \text{with} \qquad V^a = \frac{\partial \eta^a}{\partial \epsilon}\bigg|_{\epsilon=0}$$

- Ibragimov (JMAA 2006, JMAA 2007):
  - symmetries of original equation are transferred to adjoint equation
  - → application of Noether's theorem to the formal Lagrangian results in conservation laws of the extended system
  - → determination of physical conservation laws by a suitable restriction of solutions of the extended system to solutions of the original system

Nonvariational PDEs and Formal Lagrangians

- application of Noether's theorem to the formal Lagrangian results in conservation laws of the extended system

$$\operatorname{div} J(u, v) = 0$$

- self-adjointness: restrict the solution (u, v) to (u, u)

$$F^*[u, u] = \lambda F[u]$$

- quasi-self-adjointness: restrict the solution (u, v) to $(u, \phi(u))$

$$F^*[u, \phi(u)] = \lambda F[u]$$

- fixing compatible solutions of the adjoint fields by a map

$$\Phi : u \mapsto \big(u, \phi(u)\big)$$

allows us to recover conservation laws of the original system

$$\operatorname{div} J(u, \phi(u)) = 0$$

→ general framework for discrete conservation laws

Advection Equation

- linear advection equation

$$\partial_t u + c\,\partial_x u = 0 \qquad\qquad c\text{: velocity (parameter, constant)}$$

- adjoint Lagrangian with auxiliary field $v(x, t)$

$$L(v, u_x, u_t) = v\big(u_t + cu_x\big)$$

- Euler-Lagrange equations

$$\frac{\delta \mathcal{A}}{\delta v} : +u_t + cu_x = 0 \qquad \rightarrow \text{advection equation}$$

$$\frac{\delta \mathcal{A}}{\delta u} : -v_t - cv_x = 0 \qquad \rightarrow \text{adjoint equation}$$

- the adjoint equation has the same solution as the original equation

→ if u is a solution of the advection equation, then $w = (u, u)$ solves the Euler-Lagrange equations of the above adjoint Lagrangian

Ideal Compressible Euler Equations

- advective form

$$\begin{aligned} \rho v_t + \rho v \cdot \nabla v &= -\nabla p, \\ \rho_t + \nabla \cdot (\rho v) &= 0, \\ s_t + v \cdot \nabla s &= 0, \end{aligned}$$

- conservative form

$$\begin{aligned} \rho_t + \nabla \cdot (\rho v) &= 0, \\ (\rho v)_t + \nabla \cdot (\rho v v^T) &= -\nabla p, \\ (\rho \varepsilon)_t + \nabla \cdot (\rho v \varepsilon + v p) &= v \cdot \nabla p, \end{aligned}$$

- pressure form

$$\begin{aligned} \rho_t + \nabla \cdot (\rho v) &= 0, \\ (\rho v)_t + \nabla \cdot (\rho v v^T) &= -\nabla p, \\ p_t + \nabla \cdot (p v) &= -(\gamma - 1)\, p \, \nabla \cdot v, \end{aligned}$$

- skew-symmetric form

$$\begin{aligned} \rho_t + \nabla \cdot (\rho v) &= 0, \\ \tfrac{1}{2}\left[(\rho v)_t + \rho v_t\right] + \tfrac{1}{2}\left[\nabla(\rho v^2) + \rho v \cdot \nabla v\right] &= -\nabla p, \\ (\rho \varepsilon)_t + \nabla \cdot (\rho v \varepsilon + v p) &= v \cdot \nabla p, \end{aligned}$$